

A Mixed-Binary Convex Quadratic Reformulation for Box-Constrained Nonconvex Quadratic Integer Program *

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Abstract

In this paper, we propose a mixed-binary convex quadratic programming reformulation for the box-constrained nonconvex quadratic integer program and then implement IBM ILOG CPLEX 12.6 to solve the new model. Computational results demonstrate that our approach clearly outperform the very recent state-of-the-art solvers.

Keywords: Quadratic Integer Programming Convex Quadratic Programming Semidefinite Programming

1 Introduction.

We study the box-constrained nonconvex quadratic integer program:

$$\begin{aligned} \text{(P)} \quad & \min \quad x^T Q x + c^T x \\ & \text{s.t.} \quad l_i \leq x_i \leq u_i, \quad i = 1, \dots, n, \\ & \quad \quad x \in Z^n, \end{aligned}$$

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where Z^n denotes the set of n -dimensional vectors with integer entries, $l, u \in Z^n$ and $l_i < u_i$ for $i = 1, \dots, n$. Problem (P) includes the binary quadratic program as a special case (i.e., $l_i = 0$ and $u_i = 1$ for $i = 1, \dots, n$) and hence it is in general NP-hard.

When $Q \succeq 0$ (i.e., Q is positive semidefinite), the continuous relaxation is a convex program and provides an easy-to-solve lower bound of (P). In literature, there are a few solvers to globally solve (P), see [2, 3]. Moreover, some softwares developed for integer nonlinear optimization can also globally solve (P), for example, IBM ILOG CPLEX 12.6. However, CPLEX fails to solve (P) when $Q \not\succeq 0$. Actually, in this case, even the continuous relaxation of (P) remains NP-hard [7]. Very recently, in order to solve (P) with a nonconvex objective, two efficient branch-and-bound approaches based on semidefinite programming (SDP) relaxation [5] and ellipsoidal relaxation [4] are proposed, denoted by Q-MIST and GQIP, respectively. Computational results in [4] show that they outperform existing solvers such as COUENNE [1] and BARON [8].

Our contribution in this paper is to propose a mixed-binary convex quadratic programming reformulation of (P). Consequently, (P) now can be solved by IBM ILOG CPLEX 12.6. The computational results demonstrate the high efficiency of our reformulation.

The paper is organized as follows. In Section 2, we present a mixed-binary convex quadratic programming reformulation of (P). In section 3, we report the experimental results. Conclusions are made in Section 4.

Throughout the paper, $v(\cdot)$ denotes the optimal value of the problem (\cdot) . Notation $A \succeq 0$ implies that the matrix A is positive semidefinite. For a vector a , $\text{Diag}(a)$ denotes the diagonal matrix with a being its diagonal vector. $a \geq 0$ indicates that a is componentwise nonnegative. For a real value b , $\lfloor b \rfloor$ is the largest integer less than or equal to b .

2 A Mixed-Binary Convex Quadratic Reformulation

In this section, we first present a family of mixed-binary quadratic programming (MBQP) reformulations of (P) and then identify the “best” MBQP which has the tightest continuous relaxation and the objective is convex.

Define

$$I = \left\{ i \in \{1, \dots, n\} : \frac{u_i + l_i}{2} \in Z \right\}, \quad J = \left\{ i \in \{1, \dots, n\} : \frac{u_i + l_i}{2} \notin Z \right\},$$

$$m_i = \begin{cases} \frac{u_i - l_i}{2}, & \text{if } i \in I \\ u_i - l_i, & \text{if } i \in J \end{cases}, \quad \tilde{x}_i = \begin{cases} x_i - \frac{u_i + l_i}{2}, & \text{if } i \in I, \\ 2x_i - (u_i + l_i), & \text{if } i \in J. \end{cases}$$

It is trivial to verify that (P) is equivalent to

$$\begin{aligned}
(\tilde{P}) \quad & \min \quad \tilde{x}^T Q \tilde{x} + c^T \tilde{x} \\
\text{s.t.} \quad & -m_i \leq \tilde{x}_i \leq m_i, \quad \tilde{x}_i \in Z, \quad \forall i \in I, \\
& -m_i \leq \tilde{x}_i \leq m_i, \quad \frac{\tilde{x}_i - 1}{2} \in Z, \quad \forall i \in J.
\end{aligned}$$

Theorem 1 For any $\theta \in R^n$, (\tilde{P}) is equivalent to the following linearly constrained mixed-binary quadratic program in the sense that both optimal objective values are equal and both share the same optimal solution \tilde{x} :

$$\begin{aligned}
(\text{MBQP}_\theta) \quad & \min \quad \tilde{x}^T Q \tilde{x} + c^T \tilde{x} + \sum_{i \in I} \theta_i \left(\tilde{x}_i^2 - \sum_{k=1}^{m_i} k^2 y_{ik} \right) \\
& + \sum_{i \in J} \theta_i \left(\tilde{x}_i^2 - \sum_{k=0}^{(m_i-1)/2} (2k+1)^2 y_{ik} \right) \tag{1}
\end{aligned}$$

$$\text{s.t.} \quad - \sum_{k=1}^{m_i} k y_{ik} \leq \tilde{x}_i \leq \sum_{k=1}^{m_i} k y_{ik}, \quad \forall i \in I, \tag{2}$$

$$z_i \leq \sum_{k=1}^{m_i} y_{ik} \leq 1, \quad \forall i \in I, \tag{3}$$

$$\sum_{k=1}^{m_i} k y_{ik} - \tilde{x}_i \leq 2m_i z_i, \quad \forall i \in I, \tag{4}$$

$$\sum_{k=1}^{m_i} k y_{ik} + \tilde{x}_i \leq 2m_i(1 - z_i), \quad \forall i \in I, \tag{5}$$

$$- \sum_{k=0}^{(m_i-1)/2} (2k+1) y_{ik} \leq \tilde{x}_i \leq \sum_{k=0}^{(m_i-1)/2} (2k+1) y_{ik}, \quad \forall i \in J, \tag{6}$$

$$\sum_{k=0}^{(m_i-1)/2} y_{ik} = 1, \quad \forall i \in J, \tag{7}$$

$$\sum_{k=0}^{(m_i-1)/2} (2k+1) y_{ik} - \tilde{x}_i \leq 2m_i z_i, \quad \forall i \in J, \tag{8}$$

$$\sum_{k=0}^{(m_i-1)/2} (2k+1) y_{ik} + \tilde{x}_i \leq 2m_i(1 - z_i), \quad \forall i \in J, \tag{9}$$

$$y_{ik} \in \{0, 1\}, \forall i, k; \quad z \in \{0, 1\}^n. \tag{10}$$

Proof. Let \tilde{x} be any feasible solution of (\tilde{P}) . For $i = 1, \dots, n$, we define

$$\tilde{k} = \begin{cases} |\tilde{x}_i|, & \text{if } i \in I, \\ \frac{|\tilde{x}_i| - 1}{2}, & \text{if } i \in J, \end{cases} \quad \tilde{z}_i = \begin{cases} 0, & \text{if } \tilde{x}_i \geq 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then we define $\tilde{y}_{i\tilde{k}} = 1$ and $\tilde{y}_{ik} = 0$ for all $k \neq \tilde{k}$. It is not difficult to verify that $(\tilde{x}, \tilde{y}, \tilde{z})$ is a feasible solution of (MBQP_θ) with the same objective value as that of (\tilde{P}) at \tilde{x} .

Now, let $(\tilde{x}, \tilde{y}, \tilde{z})$ be a feasible solution of (MBQP_θ) .

For any $i \in I$, the constraints (2), (4)-(5) and (10) imply that either $\tilde{x}_i = \sum_{k=1}^{m_i} k\tilde{y}_{ik}$ or $\tilde{x}_i = -\sum_{k=1}^{m_i} k\tilde{y}_{ik}$. Moreover, under the constraints (3) and (10), there is at most one element in $\{\tilde{y}_{ik} : k = 1, \dots, m_i\}$ equal to 1 and the others are zeros. It follows that $\tilde{x}_i \in \{-m_i, \dots, m_i\}$ and $\tilde{x}_i^2 = \sum_{k=1}^{m_i} k^2\tilde{y}_{ik}$.

For any $i \in J$, the constraints (6), (8)-(9) and (10) imply that either $\tilde{x}_i = \sum_{k=0}^{(m_i-1)/2} (2k+1)\tilde{y}_{ik}$ or $\tilde{x}_i = -\sum_{k=0}^{(m_i-1)/2} (2k+1)\tilde{y}_{ik}$. Furthermore, under the constraints (6) and (7), there is exactly one element in $\{\tilde{y}_{ik} : k = 0, \dots, (m_i-1)/2\}$ equal to 1 and the others are zeros. It follows that $\tilde{x}_i \in \{-m_i, -m_i+2, \dots, m_i-2, m_i\}$ and $\tilde{x}_i^2 = \sum_{k=0}^{(m_i-1)/2} (2k+1)^2\tilde{y}_{ik}$.

Therefore, \tilde{x} is a feasible solution of (\tilde{P}) with the same objective value as that of (MBQP_θ) at $(\tilde{x}, \tilde{y}, \tilde{z})$. The proof is complete. \square

Remark 1 We notice that (MBQP_θ) is still equivalent to (\tilde{P}) without the constraints in the left-hand side of (3). However, these constraints help us to reduce many feasible solutions such as (x, y, z) with $y_{i1} = \dots = y_{im_i} = 0$ and $z_i = 1$ for $i \in I$.

Remark 2 A natural representation of $\{x_i : l_i \leq x_i \leq u_i, x_i \in Z\}$ may be

$$\left\{ x_i = \sum_{k=l_i}^{u_i} ky_{ik} : \sum_{k=l_i}^{u_i} y_{ik} = 1, y_{ik} \in \{0, 1\}^{u_i-l_i+1} \right\}.$$

But it need $u_i - l_i + 1$ additional binary variables y_k . In our reformulation (MBQP_θ) , we introduce an additional binary variable z_i and $\frac{u_i-l_i}{2}$ or $\frac{u_i-l_i+1}{2}$ additional binary variables y_{ik} depending on whether $i \in I$ or $i \in J$.

The choice of θ plays a great role in solving (MBQP_θ) . For any θ such that $Q + \text{Diag}(\theta) \succeq 0$, (MBQP_θ) has a convex objective function and hence can be solved by IBM ILOG CPLEX. A “best” choice of θ , denoted by θ^* , seems to be the one that maximizes the continuous relaxation of (MBQP_θ) . For convenience, we rewrite the continuous relaxation of (MBQP_θ) as

$$\begin{aligned} \text{R}(\theta) \quad & \min \quad \tilde{x}^T(Q + \text{Diag}(\theta))\tilde{x} + c^T\tilde{x} - L(\theta)^T y \\ & \text{s.t.} \quad A\tilde{x} + By \leq a, \end{aligned}$$

where $y = (y_{ik}, z_i)$, a , A, B are vectors/matrices of appropriate dimension and $L(\theta)$ is a linear

operator of θ . Moreover, since $Q + \text{Diag}(\theta) \succeq 0$, according to strong duality, we have

$$\begin{aligned}
& v(\mathbf{R}(\theta)) \\
&= \min_{x,y} \max_{\lambda \geq 0} \{ \tilde{x}^T (Q + \text{Diag}(\theta)) \tilde{x} + c^T \tilde{x} - L(\theta)^T y - \lambda^T (a - A\tilde{x} - By) \} \\
&= \max_{\lambda \geq 0} \min_{x,y} \{ \tilde{x}^T (Q + \text{Diag}(\theta)) \tilde{x} + (c + A^T \lambda)^T \tilde{x} - (L(\theta) - B^T \lambda)^T y - a^T \lambda \} \\
&= \max_{\lambda \geq 0} \tau - a^T \lambda \\
&\text{s.t.} \quad \begin{bmatrix} -\tau & \frac{1}{2}(c + A^T \lambda)^T & \frac{1}{2}(-L(\theta) + B^T \lambda)^T \\ \frac{1}{2}(c + A^T \lambda) & Q + \text{Diag}(\theta) & 0 \\ \frac{1}{2}(-L(\theta) + B^T \lambda) & 0 & 0 \end{bmatrix} \succeq 0. \tag{11}
\end{aligned}$$

Consequently, θ^* is obtained by solving an SDP:

$$\theta^* = \arg \max_{Q + \text{Diag}(\theta) \succeq 0} v(\mathbf{R}(\theta)) = \max_{\lambda \geq 0, (11)} \{ \tau - a^T \lambda \}, \tag{12}$$

where $Q + \text{Diag}(\theta) \succeq 0$ is removed since it is already implied from (11).

3 Experimental Results

In this section, we report computational results of (MBQP_{θ^*}) , where θ^* is obtained by solving the SDP (12) using SEDUMI [9] within CVX 1.2 [6].

We used the same test bed as in [4]. We randomly generated two kinds of testing instances, ternary instances where $l_i = -1$ and $u_i = 1$ for all $i = 1, \dots, n$ and the instances with larger domain: $l_i = -5$ and $u_i = 5$ for all $i = 1, \dots, n$. For each $n \in \{20, 30, 40, 50\}$ and each $p \in \{0, 0.1, \dots, 1\}$ (which is a parameter to control the inertia of Q), we created 10 random instances from different random seeds. That is, we totally solved 110 instances for each size n . For each instance, we first uniformly generated $\lfloor p \cdot n \rfloor$ and $n - \lfloor p \cdot n \rfloor$ random values from $[-1, 0]$ and $[0, 1]$, respectively. Let these n values be the eigenvalues of Q . Together with a random orthogonal basis of R^n , we created the Hessian matrix Q . Finally, all entries of c were uniformly generated at random from $[-1, 1]$.

All experiments were implemented in Matlab R2010a and IBM ILOG CPLEX 12.6. We set the time limit to one hour and the tolerance of CPLEX to be 10^{-6} (note that the default tolerance is 10^{-4}). As in [4], instances not solved to proven optimality (with accuracy less than 10^{-6}) within one hour are considered failures. We compare our method with the very recent two efficient solvers, Q-MIST [5] and GQIP [4]. As their codes are not available, we just take the time and nodes explored by Q-MIST and GQIP from [4] and showed in Table 1 the platform and software that are used by [4] and us, respectively. Notice that our machine is slower than they used in [4].

We compare the computational results for each dimension n in Tables 2 and 3, where the third column (SOLVED) gives the number of instances solved to proven optimality within one hour, out

Table 1: Platform and software used for comparison.

Platform	Q-MIST and GQIP	MBQP $_{\theta^*}$
Processor	Intel Core2 (3.2GHz)	Pentium Dual-Core (2.93GHz)
Operating System	Linux	Windows XP
Memory Size	4 GB	2 GB
Software Used	C++, Fortran 90	Matlab R2010a, CPLEX 12.6
Precision	10^{-6}	10^{-6}

of all 110 instances, the last three columns (MAX TIME, AVG TIME, AVG #NODES) list the maximum time, average time, and average number of nodes explored for the successfully solved instances, respectively.

According to Tables 2 and 3, our method has the similar computational performance as Q-MIST and GQIP for the ternary instances with $n = 20, 30, 40$, but highly outperforms Q-MIST and GQIP for the ternary instances with $n = 50$ and for the instances with the variable domain $\{-5, \dots, 5\}$. We also see that the average numbers of nodes explored by CPLEX for solving (MBQP $_{\theta^*}$) lie between those of Q-MIST and GQIP. These observations are reasonable as the continuous relaxations of the subproblems of (MBQP $_{\theta^*}$) are linearly constrained convex quadratic programs (CQP), which are weaker than those of Q-MIST (which are SDPs, less efficient to solve than CQP), but generally tighter than those of GQIP (which are nonconvex quadratic programs with an ellipsoidal constraint).

4 Conclusions

In this paper, we propose a family of mixed-binary quadratic programming reformulations for the box-constrained nonconvex quadratic integer program, denoted by (MBQP $_{\theta}$), where θ is a parameter vector. A “best” choice of θ , denoted by θ^* , is set as the one that maximizes the continuous relaxation of (MBQP $_{\theta}$). It turns out that θ^* can be obtained by solving a semidefinite program (SDP). Interestingly, (MBQP $_{\theta^*}$) has a convex quadratic objective and hence can be solved by IBM ILOG CPLEX 12.6. Computational results demonstrate that for instances with large n or large variable domain, (MBQP $_{\theta^*}$) highly outperforms the recent efficient solvers, Q-MIST [5] and GQIP [4], which are branch-and-bound approaches based on SDP relaxation and ellipsoidal relaxation, respectively.

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Table 2: Comparative results for ternary instances.

n	ALG	SOLVED	MAX TIME	AVG TIME	AVG #NODES
20	Q-MIST	110	1.0	0.2	53.5
	GQIP	110	0.6	0.1	1914.5
	MBQP _{θ^*}	110	1.4	0.1	82.6
30	Q-MIST	110	11.0	2.0	199.7
	GQIP	110	1.1	0.5	16435.7
	MBQP _{θ^*}	110	4.2	0.9	453.0
40	Q-MIST	110	105.0	15.9	831.6
	GQIP	110	10.0	1.8	307492.7
	MBQP _{θ^*}	110	5.9	2.1	2115.3
50	Q-MIST	110	1573.0	184.0	5463.7
	GQIP	110	306.7	31.6	8614806.8
	MBQP _{θ^*}	110	22.5	5.2	11204.9

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Table 3: Comparative results for instances with variable domain $\{-5, \dots, 5\}$.

n	ALG	SOLVED	MAX TIME	AVG TIME	AVG #NODES
20	Q-MIST	110	6.0	0.8	138.6
	GQIP	110	56.9	1.9	1568914.8
	MBQP $_{\theta^*}$	110	0.7	0.2	251.4
30	Q-MIST	110	237.0	17.8	1115.5
	GQIP	103	3175.1	256.6	151889362.6
	MBQP $_{\theta^*}$	110	12.9	1.0	1840.0
40	Q-MIST	109	2431.0	211.3	5861.5
	GQIP	32	3501.8	600.6	254751492.5
	MBQP $_{\theta^*}$	110	128.4	9.3	15971.7

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